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# Cauchy Inequality and Uncertainty Relations for Mixed States

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Cauchy inequality (CI) relates scalar products of two vectors and their norms. I point out other similar inequalities (SI). Starting with CI Schroedinger (1930) derived his uncertainty relation (UR). By using SI other various UR can be obtained. It is shown that they follow from the Schroedinger UR. Two generalizations of CI are obtained for mixed states described by density matrices. Using them two generalizations of UR for mixed states are derived. Both differ from the UR generalization known in the literature.The discussion of these generalizations is given.

**KEY WORDS:** quantum mechanics; uncertainty relation; density matrix.

## 1. INTRODUCTION

Heisenberg uncertainty relation (HUR) $\sigma_x^2\sigma_y^2 \geq \hbar^2/4$ is the most known example of the quantum inequalities, which are considered here. It is the corollary of quantum postulates. The main postulate is the description of physical system states by vectors of linear space with a scalar product (e.g., Hilbert space). For two vectors $\alpha_1$ and $\alpha_2$, one can also derive from these postulates the known Cauchy–Bunyakowskii–Schwarz inequality $(\alpha_1, \alpha_1)(\alpha_2, \alpha_2) \geq |(\alpha_1, \alpha_2)|^2$, which will be called the Cauchy inequality (CI).

HUR restricts possible values of dispersions of observables. CI can also be given a physical sense: it is a premise for the probability interpretation of state vectors, see Sec. 2 below.

Robertson (1930) and Schrödinger (1930) pointed out an uncertainty relation, which is more general than HUR. It is usually called the Schrödinger uncertainty relation (SUR) (see Dodonov and Man'ko, 1987; Sukhanov, 2001; Trifonov, 2002). Schrödinger (1930) derived SUR starting with CI and then obtained HUR as a particular case of SUR.

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Various modifications and generalization of CI and uncertainty relations (UR) are known. For example, there exist inequalities which contain three and more state vectors or observables (e.g., see Dodonov and Man'ko, 1987; Shirokov, 2003; Trifonov, 2002). Here we consider CI for two states and UR for two observables, the states being described either by vectors (pure states) or by density matrices (mixed states).

It is pointed out in Sec. 2 that along with CI many similar inequalities (SI) can be written which turn out to be particular cases of CI. Starting with one of these SI, one can obtain HUR. From another SI one can derive other useful UR, see Sec. 2.3.

Two generalizations of CI are obtained in Sec. 3 for the case of two mixed states described by density matrices $W_1$ and $W_2$. Both turn into CI in the case of pure states. One of them allows one to introduce the notion "the probability to find the mixed state $W_1$ in the mixed state $W_2$".

The generalization of UR for the mixed state $W$ is known, see, e.g., the review (Dodonov and Man'ko, 1987). Two new generalizations are derived in Sec. 4. All these three generalizations are different consequences of quantum postulates. All turn into the same SUR, when $W$ describes a pure state. For further discussion of these generalizations see the concluding Sec. 5.

## 2. CAUCHY INEQUALITY IN QUANTUM MECHANICS AND UNCERTAINTY RELATIONS

**2.1.** Cauchy inequality (CI) follows from the postulates of the linear space of vectors $\alpha, \beta, \ldots$ with a scalar product $(\alpha, \beta)$, e.g., see (Bohm, 1986; Fano, 1971). Further, the following postulates are used

$$\begin{aligned} &(a)\ \ (\alpha, \xi\beta) = \xi(\alpha, \beta), \quad (\xi\alpha, \beta) = \xi^*(\alpha, \beta); \\ &(b)\ \ (\alpha, \beta_1 + \beta_2) = (\alpha, \beta_1) + (\alpha, \beta_2); \\ &(c)\ \ (\alpha, \beta) = (\beta, \alpha)^*; \\ &(d)\ \ (\alpha, \alpha) \geq 0, \quad \forall\alpha. \end{aligned}$$

Here $\xi$ is a complex number. The property $(d)$ must hold for any two vectors $\alpha_1$ and $\alpha_2$ and for their superposition $\alpha_1 + \xi\alpha_2$ with an arbitrary $\xi$. Let us call this particular property postulate $(d_2)$. Cauchy inequality may be derived from $(a) - (d_2)$ in the following way (e.g., see Fano, 1971). It is easy to verify that when $\xi = -(\alpha_2, \alpha_1)/(\alpha_2, \alpha_2)$, we have

$$(\alpha_1 + \xi\alpha_2, \alpha_1 + \xi\alpha_2) = (\alpha_1, \alpha_1) - |(\alpha_1, \alpha_2)|^2/(\alpha_2, \alpha_2). \tag{1}$$

As the L.H.S. of Eq. (1) must be positive we obtain CI

$$|(\alpha_1, \alpha_2)|^2 \leq (\alpha_1, \alpha_1)(\alpha_2, \alpha_2). \tag{2}$$

The inequality can be rewritten

$$|(\alpha_1, \alpha_2)|^2/(\alpha_1, \alpha_1)(\alpha_2, \alpha_2) \le 1. \tag{3}$$

This form allows us to give the known probability interpretation to the scalar product $(\alpha_1, \alpha_2)$: the L.H.S. of (3) may be called the probability to find state $\alpha_1$ in the state $\alpha_2$ because this L.H.S. does not exceed 1, being positive (usually one supposes that $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 1$).

**2.2.** Schrödinger (1930) derived SUR from CI. Dr. O.V. Teryaev called my attention to that his derivation needs some refinement which I shall consider now.

Let $\psi$ be a state vector such that $(\psi, \psi) = 1$, and $A$ and $B$ are observables (hermitian operators). The latter may have different dimensions, e.g., $A$ has the dimension of length while $B$ is momentum. Then vectors $A\psi$ and $B\psi$ have different dimensions and cannot belong to one linear space (their sum is not defined). Meanwhile, vectors $\alpha_1$, $\alpha_2$ occurring in inequality (2) (which we are going to start with) must have the same dimension, e.g., be dimensionless. Let us assume

$$\alpha_i = d_i^{-1} \Delta A_i \psi, \quad \Delta A_i \equiv A_i - (\psi, A_i \psi), \quad i = 1, 2. \tag{4}$$

Here $d_1$ and $d_2$ are arbitrary constants which have the same dimensions as $A_1$ and $A_2$, respectively, do. The defined $\alpha_1$, $\alpha_2$ are of the same dimension as $\psi$ (e.g, are dimensionless). Substituting (4) into (2) we get

$$\left(d_1^{-1} d_2^{-1}\right)^2 |(\psi, \Delta A_1 \Delta A_2 \psi)|^2 = d_1^{-2} d_2^{-2} (\psi, (\Delta A_1)^2 \psi)(\psi, (\Delta A_2)^2 \psi). \tag{5}$$

Canceling by $d_1^{-2} d_2^{-2}$ and using the notation

$$\sigma_i^2 = (\psi, (\Delta A_i)^2 \psi) \tag{6}$$

we obtain SUR for the dispersions $\sigma_1^2$ and $\sigma_2^2$ of the observables $A_1$, $A_2$:

$$|(\psi, \Delta A_1 \Delta A_2 \psi)|^2 \le \sigma_1^2 \sigma_2^2. \tag{7}$$

In his derivation, Schrödinger omitted the above dimensional constants. In order to derive HUR from inequality (7) Schrödinger (1930) represented L.H.S. of (7) in the following manner:

$$(\psi, \Delta A_1 \Delta A_2 \psi) = (\psi, \hat{R} \psi) + (\psi, i \hat{J} \psi) \tag{8}$$

$$\hat{R} \equiv \tfrac{1}{2}\{\Delta A_1 \Delta A_2 + \Delta A_2 \Delta A_1\}, \quad i\hat{J} \equiv \tfrac{1}{2}[\Delta A_1 \Delta A_2 - \Delta A_2 \Delta A_1]. \tag{9}$$

The first term in the R.H.S. of equation (8) is the average of hermitian operator $\hat{R}$ and, therefore, is a real number $R$. The second term is imaginary and is equal to $iJ$, $J$ being real ($\hat{J}$ denotes hermitian operator). So we have $(\psi, \Delta A_1 \Delta A_2 \psi) = R + iJ$ and (7) can be represented as

$$R^2 + J^2 \le \sigma_1^2 \sigma_2^2. \tag{10}$$

If (10) holds, then we have, of course, $J^2 \leq \sigma_1^2\sigma_2^2$, i.e., HUR

$$\frac{1}{4}|(\psi, [A_1, A_2]\psi)|^2 \leq \sigma_1^2\sigma_2^2. \tag{11}$$

Let us note that, if $\sigma_1^2\sigma_2^2 = J^2$, then it follows from (10) that $R^2$ must be zero (Schrödinger, 1930).

**2.3.** In Sec. 2.1 CI has been derived from $(\alpha_1 + \xi\alpha_2, \alpha_1 + \xi\alpha_2) \geq 0$ using the special fitting of $\xi$: $\xi = -(\alpha_2, \alpha_1)/(\alpha_2, \alpha_2)$. However, the starting inequality must hold for any $\xi$. Various $\xi$ lead to various inequalities similar to CI, let us call them SI. Using the substitution (4) one may obtain from SI various UR. A natural question arises: what information these inequalities provide as compared to CI and SUR?

First of all note that, if $\xi$ differs strongly enough from the value $-(\alpha_2, \alpha_1)/(\alpha_2, \alpha_2)$, then we get from $(\alpha_1 + \xi\alpha_2, \alpha_1 + \xi\alpha_2)$ no real restrictions for $(\alpha_1, \alpha_1)$, $(\alpha_2, \alpha_2)$, $(\alpha_1, \alpha_2)$. This follows from such easily verifiable identity

$$\begin{aligned}(\alpha_1 + \xi\alpha_2, \alpha_1 + \xi\alpha_2) = (\alpha_1, \alpha_1) - (r^2 + j^2)/(\alpha_2, \alpha_2)\\ + (\alpha_2, \alpha_2)\{\rho + r/(\alpha_2, \alpha_2)\}^2\\ + (\alpha_2, \alpha_2)\{\eta - j/(\alpha_2, \alpha_2)\}^2.\end{aligned} \tag{12}$$

Here $\xi = \rho + i\eta$ and

$$r = \mathrm{Re}(\alpha_1, \alpha_2), \quad j = \mathrm{Im}(\alpha_1, \alpha_2). \tag{13}$$

Indeed, if the curly brackets in the R.H.S. of equation (12) are large enough, then the positivity of $(\alpha_1 + \xi\alpha_2, \alpha_1 + \xi\alpha_2)$ is ensured at any $(\alpha_1, \alpha_1)$, $(\alpha_2, \alpha_2)$, $(\alpha_1, \alpha_2)$.

Now let us discuss two examples of SI, which give nontrivial inequalities of physical interest. At first, consider the case $\xi = i\eta$, i.e., $\rho = 0$. Then equation (12) turns into

$$(\alpha_1 + i\eta\alpha_2, \alpha_1 + i\eta\alpha_2) = (\alpha_1, \alpha_1) - j^2/(\alpha_2, \alpha_2) + (\alpha_2, \alpha_2)\{\eta - j/(\alpha_2, \alpha_2)\}. \tag{14}$$

If $\eta - j/(\alpha_2, \alpha_2) = 0$, then it follows from $(\alpha_1 + i\eta\alpha_2, \alpha_1 + i\eta\alpha_2) \geq 0$ that

$$(\alpha_1, \alpha_1)(\alpha_2, \alpha_2) - j^2 \geq 0, \quad j = \mathrm{Im}(\alpha_1, \alpha_2). \tag{15}$$

Inequality (15) is of physical interest because one can obtain from it the HUR, see (11), using the substitution (4) (note that $\mathrm{Im}(\psi, \Delta A_1 \Delta A_2 \psi)$ has been denoted by $J$ in Sec. 2.2).

Similarly, one may treat another particular case: $\xi = \rho$, i.e., $\eta = 0$. Then one gets instead of (15)

$$(\alpha_1, \alpha_1)(\alpha_2, \alpha_2) - r^2 \geq 0, \quad r = \mathrm{Re}(\alpha_1, \alpha_2)$$

and instead of HUR the inequality

$$\frac{1}{4}|(\psi, \{\Delta A_1 \Delta A_2 + \Delta A_2 \Delta A_1\}\psi)|^2 \leq \sigma_1^2 \sigma_2^2. \tag{16}$$

This UR means that the product of dispersions may be restricted from below also in the case when observables $A_1$ and $A_2$ commute and HUR turns into the inequality $\sigma_1^2\sigma_2^2 \geq 0$ which is trivial: dispersions are positive by definition, see equation (6).

Let us stress that CI is not only a necessary consequence of the postulate $(d_2)$ but also is sufficient: the postulate $(d_2)$, i.e., $(\alpha_1 + \xi\alpha_2, \alpha_1 + \xi\alpha_2) \geq 0$, follows from CI because squares of the curly brackets in equation (12) are positive at any $\xi$. Meanwhile, SI are not sufficient (being necessary, of course). For example, let (15) hold. Then the R.H.S. of equation (12) may turn out to be negative, because it contains besides nonnegative contributions also the negative one $-r^2/(\alpha_2, \alpha_2)$.

So CI is equivalent to $(d_2)$ and may replace it. Another proof of this equivalency is presented in (Shirokov, 2003), Sec. 3.1, the case $n = 2$.

**2.4.** The relation of SUR following from CI to other UR following from SI (e.g., see (11) and (16)) can be formulated using a physical language: SUR is the most restrictive inequality for dispersions. Other UR, e.g., (11) and (16), follow from SUR. For example, the region of possible values of $\sigma_1^2\sigma_2^2$ which is allowed by (11) is greater than the region allowed by SUR. In particular, when $A_1$ and $A_2$ commute inequality (11) turns into trivial one $\sigma_1^2\sigma_2^2 \geq 0$, while SUR, see (10), shows that in this case $\sigma_1^2\sigma_2^2$ must be larger than a nonzero (generally) quantity $R^2$. However, HUR is simpler than SUR and, therefore, may be useful. For example, if $[A_1, A_2]$ is equal to a number $ic$, then for any normalizable $\psi$ the product $\sigma_1^2\sigma_2^2$ is restricted from below by the known constant: $c^2/4 \leq \sigma_1^2\sigma_2^2$ .

## 3. CAUCHY INEQUALITIES FOR DENSITY MATRICES

Till now physical system states were supposed to be described by vectors of linear space (e.g, Hilbert space). In this section the states are described by density matrices. Two various inequalities for two density matrices will be obtained which generalize CI, see (2).

**3.1.** The derivation of the first inequality uses the following definition of the density matrices $W_1$ and $W_2$:

$$W_i = \sum_m p_m^{(i)} \big|\omega_m^{(i)}\big\rangle\big\langle\omega_m^{(i)}\big|, \quad p_m^{(i)} > 0, \quad i = 1, 2, \tag{17}$$

where $\omega_m$ are state vectors and $p_m$ are their weights. The orthogonality of $\omega_m$ is not supposed. I do not suppose also that $(\omega_m, \omega_m) = 1$ and that $\mathrm{Sp}W_i = 1$, as is assumed by Messiah (1961) v. 1, chapter VIII.21. Such a general definition of $W_i$

will be needed below in Sec. 4. One has

$$\mathrm{Sp}W_i = \sum_m p_m^{(i)} \langle \omega_m^{(i)} | \omega_m^{(i)} \rangle, \tag{18}$$

$$\mathrm{Sp}W_1 W_2 = \sum_{m,n} p_m^{(1)} p_n^{(2)} \left| \langle \omega_m^{(1)} | \omega_n^{(2)} \rangle \right|^2. \tag{19}$$

Taking into account CI

$$\left| \langle \omega_m^{(1)} | \omega_n^{(2)} \rangle \right|^2 \leq \langle \omega_m^{(1)} | \omega_m^{(1)} \rangle \langle \omega_n^{(2)} | \omega_n^{(2)} \rangle$$

and Eq. (18) one gets

$$\mathrm{Sp}W_1 W_2 \leq \mathrm{Sp}W_1 \mathrm{Sp}W_2. \tag{20}$$

Equality in (20) is attained, when vectors $\omega_m^{(1)}$ are parallel to vectors $\omega_n^{(2)}$, $\forall\, m, n$. In particular, all $\omega_m^{(1)}$ must be parallel to one vector, e.g., $\omega_1^{(2)}$. This means that vectors $\omega_m^{(1)}$ must be parallel to each other: $\omega_m^{(1)} \| \omega_m^{(1)} \| \omega_m^{(1)} \| \ldots$. Analogously $\omega_n^{(2)}$ must be parallel pairwise. However, $W_1$ and $W_2$ then describe pure states, otherwise one has

$$0 \leq \mathrm{Sp}W_1 W_2 < \mathrm{Sp}W_1 \mathrm{Sp}W_2$$

(the positivity of $\mathrm{Sp}W_1 W_2$ follows from Eq. (19)). So the ratio $\mathrm{Sp}W_1 W_2 / \mathrm{Sp}W_1 \mathrm{Sp}W_2$ being positive cannot be equal to unity if both states $W_1$ and $W_2$ are mixed. This circumstance does not allow one to treat the ratio as "the probability to find the state $W_1$ in the state $W_2$." Indeed, it is natural to expect that such a probability is unity when $W_1 = W_2$ but actually we have $\mathrm{Sp}(W_i)^2 / (\mathrm{Sp}W_i)^2 < 1$ for a mixed state $W_i$.

In the particular case of pure states $W_i = |\alpha_i\rangle\langle\alpha_i|$ inequality (20) turns into CI, see (2). So (20) may be considered, as the generalization of CI.

**3.2.** To obtain another inequality for $W_1$ and $W_2$, let us consider the linear space of hermitian matrices and their superpositions $Z$ with complex coefficients. The scalar product of elements $Z_1$ and $Z_2$ (vectors) of the space is defined as the number $\mathrm{Sp}Z_1^\dagger Z_2$ († means hermitian conjugation). One can verify that the product has the properties

$$\mathrm{Sp}Z_1^\dagger Z_2 = (\mathrm{Sp}Z_2^\dagger Z_1)^*, \tag{21}$$

$$\mathrm{Sp}Z^\dagger Z \geq 0, \tag{22}$$

i.e., postulates ($c$) and ($d$) (see Sec. 2) hold. Let us stress that the matrices $Z$ play the role of vectors of the space.

Let us substitute in inequality (22) the superposition $Z = \sum_i \mu_i W_i$, $i = 1, 2, \ldots, n$, where $W_i$ are (hermitian) density matrices and $\mu_i$ may be any complex numbers. One has

$$\mathrm{Sp} Z^\dagger Z = \sum_{i,j} \mu_i^* \mu_j \mathrm{Sp} W_i W_j. \tag{23}$$

The R.H.S. of Eq. (23) is of the form $\mu^\dagger M \mu = \sum_{i,j} \mu_i^* M_{ij} \mu_j$, where $\mu$ is an arbitrary $n$-vector with components $\mu_i$ and $M$ is the $n \times n$ matrix with elements $M_{ij} = \mathrm{Sp} W_i W_j$. The L.H.S. of Eq. (23) is nonnegative and, therefore, $M$ must be a positively (more exactly nonnegatively) defined matrix. The necessary and sufficient condition of this property is nonnegativity of all principal minors of $M$ (e.g., see Gantmakher, 1960, chapter 10.4, Theorem 4). The minors of the first order are equal to $\mathrm{Sp} W_i^2$, $i = 1, 2, \ldots, n$ and they are nonnegative, see Eq. (19). The nonnegativity of the second order minors gives in particular

$$[\mathrm{Sp}(W_1 W_2)]^2 \leq \mathrm{Sp} W_1^2 \mathrm{Sp} W_2^2 \tag{24}$$

or

$$0 \leq [\mathrm{Sp}(W_1 W_2)]^2 \big/ \mathrm{Sp} W_1^2 \mathrm{Sp} W_2^2 \leq 1. \tag{25}$$

When $W_i = |\alpha_i\rangle\langle\alpha_i|$ inequality (24) turns into "squared" CI

$$|(\alpha_1, \alpha_2)|^4 \leq (\alpha_1, \alpha_1)^2 (\alpha_2, \alpha_2)^2$$

which is equivalent to CI (see (2)) itself.

Unlike the above discussed ratio $\mathrm{Sp}(W_1 W_2)/\mathrm{Sp} W_1 \mathrm{Sp} W_2$ the ratio $[\mathrm{Sp}(W_1 W_2)]^2/\mathrm{Sp} W_1^2 \mathrm{Sp} W_2^2$ occurring in (25) turns into unity, if $W_1 = W_2$ (even if the state is mixed). This allows one to suggest the notion "the probability $\rho_{12}$ to find the state $W_1$ in the state $W_2$" defining it as

$$\rho_{12} = \mathrm{Sp}(W_1 W_2) \big/ \left[\mathrm{Sp} W_1^2 \mathrm{Sp} W_2^2\right]^{1/2}. \tag{26}$$

This quantity turns into the known probability $|(\alpha_1, \alpha_2)|^2/(\alpha_1, \alpha_1)(\alpha_2, \alpha_2)$ when $W_1 = |\alpha_1\rangle\langle\alpha_1|$ and $W_2 = |\alpha_2\rangle\langle\alpha_2|$, see Sec. 2.1.

## 4. UNCERTAINTY RELATIONS FOR MIXED STATES

In Sec. 2, uncertainty relations for pure states were considered. Here I shall deal with various extensions of UR for the case of a mixed state described by a density matrix. I call them mixed state uncertainty relations MUR.

**4.1.** The first extension MUR 1 follows from the generalization (20) of CI to the case of mixed states. Consider such density matrices

$$W_i = \Delta A_i W \Delta A_i, \quad \Delta A_i \equiv A_i - \mathrm{Sp} W A_i, \quad i = 1, 2 \tag{27}$$

$$W = \sum_m p_m |\omega_m\rangle\langle\omega_m|$$

(cf. Eq. (4), dimension constants $d_i$ are omitted here). I assume that $(\omega_m, \omega_m) = 1$ and $\mathrm{Sp}W = 1$, see (Messiah, 1961), chapter VIII.21. However, the vectors $\Delta A_i \omega_m$ are not then normalized. Density matrices $W_1$ and $W_2$ defined by Eq. (27)

$$W_i = \sum_m p_m |\Delta A_i \omega_m\rangle\langle\Delta A_i \omega_m|$$

are particular cases of those used in Sec. 3, Eq. (17). The spurs of $W_1$ and $W_2$ are not supposed to be equal to unity.

Substituting $W_i$ defined by equation (27) in inequality (20) one obtains

$$\mathrm{Sp}\Delta A_1 W \Delta A_1 \Delta A_2 W \Delta A_2 < \mathrm{Sp}(\Delta A_1)^2 W \mathrm{Sp}(\Delta A_2)^2 W. \quad (28)$$

Here $\mathrm{Sp}(\Delta A)^2 W$ is the dispersion of the observable $A$ in the mixed state $W$. One gets SUR, see (7), when $W$ describes a pure state: $W = |\psi\rangle\langle\psi|$.

Let us obtain from (28) an inequality resembling HUR. For this purpose represent the L.H.S. of (28) as

$$\begin{aligned} \mathrm{Sp}W\Delta A_1 \Delta A_2 W \Delta A_2 \Delta A_1 &= \mathrm{Sp}W(\hat{R} + i\hat{J})W(\hat{R} - i\hat{J}) \\ &= \mathrm{Sp}W\hat{R}W\hat{R} + \mathrm{Sp}W\hat{J}W\hat{J} \end{aligned} \quad (29)$$

for $\hat{R}$ and $\hat{J}$, see equation (9). Using equation (27) one can show that $\mathrm{Sp}W\hat{R}W\hat{R}$ and $\mathrm{Sp}W\hat{J}W\hat{J}$ are nonnegative numbers (because $\hat{R}$ and $\hat{J}$ are hermitian). Let us denote them as $r^2$ and $j^2$, respectively. Then (28) can be rewritten as

$$r^2 + j^2 \leq \sigma_1^2 \sigma_2^2. \quad (30)$$

Inequality (30) resembles (10) but has different positive numbers in its L.H.S.

If (30) holds, then, of course, $j^2 \leq \sigma_1^2\sigma_2^2$, $j^2 \equiv \mathrm{Sp}W\hat{J}W\hat{J}$. If the commutator $[A_1, A_2]$ is equal to the number $ic$, then $\hat{J} = c/2$ and $j^2 = \frac{1}{4}c^2\mathrm{Sp}W^2$. One obtains the inequality resembling HUR

$$\tfrac{1}{4}c^2 \mathrm{Sp}(W^2) \leq \sigma_1^2 \sigma_2^2. \quad (31)$$

**4.2.** Using the substitution (27), one obtains from another generalization (24) of CI the following extension of UR:

$$[\mathrm{Sp}\Delta A_1 W \Delta A_1 \Delta A_2 W \Delta A_2]^2 \leq [\mathrm{Sp}(\Delta A_1 W \Delta A_1)^2][\mathrm{Sp}(\Delta A_2 W \Delta A_2)^2]. \quad (32)$$

This extension, MUR 2, is more cumbersome than the previous one. In the particular case $W = |\psi\rangle\langle\psi|$, inequality (32) turns into the "squared" SUR

$$|(\psi, \Delta A_1 \Delta A_2 \psi)|^4 \leq \sigma_1^4 \sigma_2^4$$

which is equivalent to SUR, see (7).

Inequality (32) does not contain dispersions. However, using the inequalities $\mathrm{Sp}(W_i^2) \leq [\mathrm{Sp}W_i]^2$ for the density matrices $W_i$, see Eq. (27), one obtains from (32) the relaxed inequality

$$[\mathrm{Sp}\Delta A_1 W \Delta A_1 \Delta A_2 W \Delta A_2]^2 \leq \sigma_1^4 \sigma_2^4 \quad (33)$$

which contains dispersions. It coincides with the "squared" inequality (28), i.e., with (28) itself. As far as the relaxation of (32) coincides with (28), one may conclude that inequality (32) itself is more restrictive than (28).

**4.3.** Let us set forth the derivation of the known generalization of SUR for a mixed state, cf. (Bohm, 1986), chapter II.6; (Dodonov and Man'ko, 1987), chapter 2. I call it MUR 3.

Consider linear space of hermitian matrices $A_i$ (observables) and their superpositions $Z = \sum_i \mu_i A_i$ with complex coefficients $\mu_i$. The matrices play the role of vectors of this space. Their scalar product $(Z_1, Z_2)$ is defined in a different way than in Sec. 3.2.

$$(Z_1, Z_2) \equiv \mathrm{Sp} W Z_1^\dagger Z_2.$$

It has the properties $(Z_1, Z_2) = (Z_2, Z_1)^*$ and

$$(Z, Z) = \mathrm{Sp} W Z^\dagger Z \geq 0, \quad \forall Z. \tag{34}$$

Inserting $Z = \sum_i \mu_i \Delta A_i$, $i = 1, 2$ into (34) one gets

$$\mathrm{Sp} W Z^\dagger Z = \sum_{i,j} \mu_i^* \mu_j \mathrm{Sp} W \Delta A_i \Delta A_j. \tag{35}$$

Due to (34) the R.H.S. of (35) must be nonnegative. The necessary and sufficient condition of this property is the nonnegativity of principle minors of the $2 \times 2$ matrix with the elements $\mathrm{Sp} W \Delta A_i \Delta A_j$, cf. Sec. 3.2. In particular, the nonnegativity of the second order minor gives the desired MUR

$$|\mathrm{Sp} W \Delta A_1 \Delta A_2|^2 \leq \mathrm{Sp}(\Delta A_1)^2 W \mathrm{Sp}(\Delta A_2)^2 W \tag{36}$$

(the property $\mathrm{Sp} \Delta A_1 W \Delta A_2 = (\mathrm{Sp} \Delta A_2 W \Delta A_1)^*$ was used). The inequality (36) turns into SUR, see (7) if $W = |\psi\rangle\langle\psi|$.

The obtained MUR can be represented as

$$R^2 + J^2 \leq \sigma_1^2 \sigma_2^2; \quad R = \mathrm{Sp} W \hat{R}, \quad J = \mathrm{Sp} W \hat{J} \tag{37}$$

for $\hat{R}$ and $\hat{J}$, see Eq. (9) (one must repeat the computations performed in Sec. 2.2 substituting $\mathrm{Sp} W \ldots$ for $(\psi, \ldots \psi)$). Inequality (37) is analogous to (30). If $[A_1, A_2]$ is equal to the number $ic$, one obtains from (37) omitting $R^2$

$$\frac{1}{4} c^2 (\mathrm{Sp} W)^2 \leq \sigma_1^2 \sigma_2^2. \tag{38}$$

This inequality coincides with HUR for a normalized pure state, i.e., $c^2/4 \leq \sigma_1^2 \sigma_2^2$, if the mixed state $W$ is also normalized: $\mathrm{Sp} W = 1$. As compared to (31) it is more restrictive because $\mathrm{Sp}(W^2) \leq (\mathrm{Sp} W)^2 = 1$.

The obtained inequalities MUR 1, (28); MUR 2, (32), and MUR 3, (36), will be discussed in Conclusion.

## 5. CONCLUSION

The known Cauchy inequality (CI) follows from postulates of linear space of state vectors. Starting with CI Schrödinger (1930) derived the uncertainty relation, named here SUR.

Besides CI, other similar inequalities (SI) follow from the mentioned postulates (for their definition see Sec. 2). They also are of interest because they allow one to obtain some UR, e.g., Heisenberg uncertainty relation (HUR). However, these UR are less restrictive (informative) than SUR, being special cases of SUR.

All UR have a physical sense giving restrictions on dispersions of observables. Cauchy inequality also may be given a physical sense: it is the premise of the known probability interpretation of state vectors, see Sec. 2.1.

In Sec. 2, CI and UR for pure states (described by vectors) are discussed. Two generalizations of CI to the case of mixed states (described by density matrices) are obtained in Sec. 3. Both turn into the usual CI when the states are pure.

Starting with these generalizations of CI I have derived in Sec. 4 two extensions of UR for mixed states (named MUR 1 and MUR 2). This was done by means of the approach which Schrödinger (1930) used for UR derivation from CI in the case of pure states. The derivation of the extension of UR for mixed states which is known in the literature (named MUR 3), does not use any CI, see Sec. 4.3.

So starting with the quantum postulates three UR for mixed states can be obtained. All are necessary corollaries of the postulates. The corollaries are different; e.g., MUR 1 and MUR 3 contain dispersions of observables while MUR 2 does not; the L.H.S. of MUR 3, see (36), contains $\mathrm{Sp}\Delta A_1 W \Delta A_2$ which is linear in $W$ while the L.H.S. of MUR 1, see (28), and MUR 2, see (32), contain Spur which quadratically depends upon $W$.

All these UR may be called generalizations of UR for mixed states because they turn into the same SUR when the state is pure.

It was shown in Sec. 2 that SUR is the most restrictive UR in the case of pure states. When the states are mixed MUR 3 seems to be more restrictive than MUR 1: compare (31) with (38). It was also concluded in Sec. 4.2 that MUR 2 is more restrictive than MUR 1.

One may prefer MUR 3 because it has the following merit: quantities entering it have the known physical sense: $\sigma_i^2$ are dispersions, and $\mathrm{Sp}W\hat{R}$ and $\mathrm{Sp}W\hat{J}$ are averages of the observables (hermitian operators) $\hat{R}$ and $\hat{J}$, see equation (9). Meanwhile MUR 2, see (32), does not contain dispersions. Besides, both MUR 1 and MUR 2 contain spurs which depend on $W$ quadratically. Mathematical means of quantum mechanics allow one to calculate such quantities. However, their physical interpretation is unknown and still must be devised. For example, one may suggest to interpret $\mathrm{Sp}W^2$ as a measure of "mixity" of the state $W$ ($\mathrm{Sp}W^2$ being unity when the state is pure).

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